Biclosed Relations with respect to Torsion Theories

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torsion theories

Definition [Dickson, 1966]

A torsion theory is a pair of classes $(\mathbb{T}, \mathbb{F})$ in R-Mod such that:

- $\mathbb{T} \cap \mathbb{F} = \{0\}$.
- $\mathbb{T}$ is closed under epimorphisms, $\mathbb{F}$ is closed under monomorphisms.
- For each $M \in R$-Mod there exist $T \in \mathbb{T}$, $F \in \mathbb{F}$, and an exact sequence:

$$0 \to T \to M \to F \to 0$$

torsion theories

Characterization [Dickson, 1966]

A pair of classes $(\mathbb{T}, \mathbb{F})$ in R-Mod is a torsion theory if, and only if:

- $\mathbb{T} = l(\mathbb{F}) := \{M \mid \forall N \in \mathbb{F}, Hom_R(M, N) = 0\}$
- $\mathbb{F} = r(\mathbb{T}) := \{N \mid \forall M \in \mathbb{T}, Hom_R(M, N) = 0\}$

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Notice that $\langle r, l \rangle : \wp(R\text{-Mod}) \to \wp(R\text{-Mod})$ is an antitone Galois connection, i.e.:

- r, l are order-reversing.
- $l \circ r, r \circ l$ are inflationary.

correspondence polarities-relations

Proposition

There is a one-to-one correspondence between relations $\mathcal{S} \subseteq A \times B$ and Galois connections $\langle f, g \rangle : \wp(A) \to \wp(B)$:

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- Given $\mathcal{S} \subseteq A \times B$:
  - $f_{\mathcal{S}}(U) := \bigcap_{a \in U} a\mathcal{S}$, where $a\mathcal{S} := \{b \in B \mid (a, b) \in \mathcal{S}\}$.
  - $f^{\mathcal{S}}(V) := \bigcap_{b \in V} \mathcal{S}b$, where $\mathcal{S}b := \{a \in A \mid (a, b) \in \mathcal{S}\}$.

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- Given $\langle f, g \rangle : \wp(A) \to \wp(B)$:

$$\begin{aligned} \mathcal{S}_{\langle f,g \rangle} &:= \{(a, b) \in A \times B \mid b \in f(\{a\})\} \\ &= \{(a, b) \in A \times B \mid a \in g(\{b\})\}. \end{aligned}$$

biclosed relations

Definition

Given a set A, a Moore family is a family Φ of subsets of A which satisfy $\cap\Phi' \in \Phi$ for every $\Phi' \subseteq \Phi$.

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Definition [Domenach, Leclerc, 2000]

Given Moore families $\Phi \subseteq \wp(A)$ and $\Phi' \subseteq \wp(B)$,
a relation $\mathcal{R} \subseteq A \times B$ is called biclosed wrt (Φ, Φ') if:

- $\forall\, a \in A, a\mathcal{R} \in \Phi'$
- $\forall\, b \in B, \mathcal{R}b \in \Phi$

biclosed relations

some Moore situations

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- $\Phi = cls(\varphi), \Phi' = cls(\varphi')$, are Moore families, where φ, φ' are any closure operators on $\wp(A), \wp(B)$, respectively.

biclosed relations

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- $\Phi = cls(\varphi), \Phi' = cls(\varphi')$, are Moore families, where φ, φ' are any closure operators on $\wp(A), \wp(B)$, respectively.
- $\varphi = g \circ f, \varphi' = f \circ g$ are closure operators, where $\langle f, g \rangle : \wp(A) \to \wp(B)$ is any Galois connection. In this case $\Phi = Im(g)$ and $\Phi' = Im(f)$.

biclosed relations

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- In case that $\langle f, g \rangle : \wp(A) \to \wp(B)$ is the Galois connection induced by any relation $\mathcal{S} \subseteq A \times B$:

biclosed relations

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- In case that $\langle r, l\rangle : \wp(R\text{-Mod}) \to \wp(R\text{-Mod})$ is the Galois connection induced by the relation $\mathcal{H} = \{(M, N) \mid Hom_R(M, N) = 0\}$:

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  - $\Phi = \{\mathbb{T} \mid \mathbb{T}$ is a torsion class$\}$
  - $\Phi' = \{\mathbb{F} \mid \mathbb{F}$ is a torsion-free class$\}$

biclosed relations wrt torsion theories

Characterization

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It is a (big) complete lattice, where:

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product and coproduct of preradicals

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R-radid denotes the class of all idempotent radicals.

$\mathcal{R}$-torsion theories and idempotent radicals

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If $\mathcal{R}$ is a biclosed relation wrt $\mathcal{H}$ then:

- $\langle \lambda_{\mathcal{R}}, \mu_{\mathcal{R}} \rangle$ is an isotone Galois connection on *R*-radid:

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$$\mu_{\mathcal{R}}(\tau) := \text{Tr}_{\,f^{\mathcal{R}}(\mathbb{F}_\tau)} = \bigvee_{M \in f^{\mathcal{R}}(\mathbb{F}_\tau)} \alpha_M^M$$

- It induces an isomorphism between the intervals:

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- It induces an isomorphism between the intervals:

$$[[\sigma_{\mathcal{R}}, 1]] := \{\sigma \in R\text{-radid} \mid \mathbb{T}_\sigma \in (f)_{\mathcal{R}}\text{-}cls\},$$
$$[[0, \tau_{\mathcal{R}}]] := \{\tau \in R\text{-radid} \mid \mathbb{F}_\tau \in cls\text{-}(f)_{\mathcal{R}}\}.$$

continuous and cocontinuous functors

Characterization

Let $H : \mathcal{A} \to \mathcal{B}$ be a covariant functor between bicomplete abelian categories.

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examples: continuous and cocontinuous functors

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- $\mathrm{Hom}_R(M, \_)$ ($\mathrm{Hom}_R(\_, M)$) is an almost continuous covariant (contravariant) functor.

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- Every continuous (cocontinuous) functor is almost continuous (almost cocontinuous).
- $\text{Hom}_R(M, \_)$ ($\text{Hom}_R(\_, M)$) is an almost continuous covariant (contravariant) functor.
- $M \otimes \_$ is an almost cocontinuous covariant functor.

almost continuous and cocontinuous functors

examples: the other way round

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- $M \otimes \_$ is an almost continuous covariant functor if, and only if, M_R is flat and Mittag-Leffler.

almost continuous and cocontinuous functors

examples: the other way round

- $M \otimes \_$ is an almost continuous covariant functor if, and only if, M_R is flat and Mittag-Leffler.
- $\mathrm{Hom}_R(M, \_)$ is an almost cocontinuous covariant functor if, and only if, ${}_RM$ is projective and it has the following property: for any family $\{M_\alpha\}_{\alpha\in\Lambda}$ of R-modules, the image of every homomorphism $M \to \bigoplus_{\alpha\in\Lambda} M_\alpha$ is contained in $\bigoplus_{\alpha\in\Lambda'} M_\alpha$ for some finite subset Λ' of Λ.

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- Every left exact preradical is an almost continuous covariant functor.

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- Every left exact preradical is an almost continuous covariant functor.
- Every preradical α_I^R, where I is an ideal of R, is an almost cocontinuous covariant functor.

AC bifunctors

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- For each object N of $\mathcal{B}$, $K(\_, N)$ is an almost continuous contravariant functor.

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Proposition

If $K(\_,\_) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathcal{C}$ is an AC bifunctor, $F : \mathcal{A}' \longrightarrow \mathcal{A}$ is an almost cocontinuous covariant functor and $G : \mathcal{B}' \longrightarrow \mathcal{B}$ is an almost continuous covariant functor then:

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If $K(\_,\_) : (R\text{-Mod})^{op} \times R\text{-Mod} \longrightarrow \mathcal{A}b$ is an AC bifunctor, then $\mathcal{R}_{(K)} := \{(M,N) \in (R\text{-Mod})^2 \mid K(M,N) = 0\}$ is biclosed wrt $\mathcal{H}$.

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If $\langle F, G\rangle$ is an adjoint pair then $\mathcal{R}^F = \mathcal{R}_G$.

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AC bifunctors

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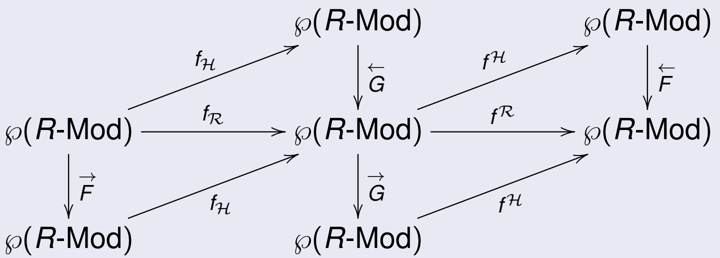
Proposition

If $\mathcal{R}$ is the biclosed relation induced by the adjoint pair $\langle F, G \rangle$ then the following diagrams commute:



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- We define a preorder on R-BiMod: $L \preceq K$ if L generates K.
- The equivalence class of L is denoted by $[L]_\sim$
- There is an order-preserving assignment:

$$\Psi : R\text{-BiMod}/\sim \; \longrightarrow [\mathcal{H}]_\preceq$$
$$\Psi([L]_\sim) := \mathcal{R}_{[L]}$$

bimodules R/I

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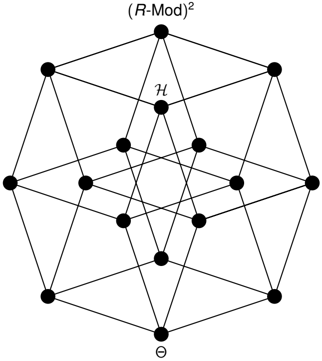
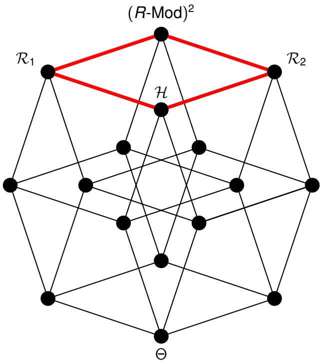
- Ψ is injective.
- $Im(\Psi) = \{\mathcal{R} \in [\mathcal{H}]_{\preceq} \mid \mathcal{H} \subseteq \mathcal{R}\}$.



The Boolean lattice $[\mathcal{H}]_{\preceq}$
for a semisimple Artinian ring R with $|R\text{-simp}| = 2$



The image of $\Psi : R\text{-BiMod}/\sim\ \longrightarrow [\mathcal{H}]_{\preceq}$
for a semisimple Artinian ring R with $|R\text{-simp}| = 2$

References

Dickson, S.E.
A torsion theory for abelian categories
Trans. Amer. Math. Soc. **121** pp. 223–235, 1966.

Domenach, F., Leclerc, B.
Biclosed binary relations and Galois connections
Order **18** pp. 89–104, 2000.

Fernández-Alonso, R., Raggi, F., Ríos, J., Rincón, H., Signoret, C.
The lattice structure of preradicals
Comm. Algebra. **30(3)** pp. 1533-1544, 2002.